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A-STABLE RUNGE–KUTTA PROCESSES

F. H. CHIPMAN

Abstract.

The concept of strong A-stability is defined. A class of strongly A-stable Runge–Kutta processes is introduced. It is also noted that several classes of implicit Runge–Kutta processes defined by Ehle [6] are A-stable.

1. Introduction.

The v-stage Runge–Kutta process for numerically solving the n-dimensional system $y' = f(x,y)$, $y(0) = y_0$, is given by

$$y_{m+1} = y_m + hWK, \quad m = 1,2,\ldots, \tag{1.1}$$

where

$$K = \begin{pmatrix} K_1 \\ \vdots \\ K_v \end{pmatrix} \quad \text{with } K_i = f\left(x_m + c_i h,\ y_m + h \sum_{j=1}^{v} b_{ij} K_j\right),$$

and $W = (w_1 I, \ldots, w_v I)$, with I the $n \times n$ identity matrix.

To study the stability of process (1.1), we apply it to the scalar equation $y' = qy$, $\mathrm{Re}(q) < 0$. The resulting difference equation is

$$y_{m+1} = y_m + hWK \tag{1.2}$$

where

$$K = \begin{pmatrix} q \\ \vdots \\ q \end{pmatrix} y_m + qh(b_{ij})K\,. \tag{1.3}$$

Definition 1.1. Process (1.1) is well defined if (1.3) has a unique solution for all q with $\mathrm{Re}(q) < 0$.

Hence, for a well defined process, (1.2) may be written

$$\begin{aligned} y_{m+1} &= (1 + qhW(I - qhB)^{-1}e)y_m \\ &= E(qh)y_m\,, \end{aligned} \tag{1.4}$$

where $e = (1 \ldots 1)^T$ and $B = (b_{ij})$.

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DEFINITION 1.2. Process (1.1) is A-stable, in the sense of Dahlquist [5], if $|E(z)| < 1$ for all complex z with negative real part.

DEFINITION 1.3. Process (1.1) is strongly A-stable if it is A-stable and $\lim_{\mathrm{Re}(z)\to-\infty} E(z) = 0$.

If a process is not well defined, then $E(z)$ is not defined at v or fewer points in the left half complex plane. Rewriting $E(z)$ as a rational function

$$E(z) = \frac{\det(I - zB) + zW \operatorname{adj}(I - zB)e}{\det(I - zB)} = \frac{P(z)}{Q(z)},$$

we see that an A-stable process is well defined if $P(z)$ and $Q(z)$ have no common zeros in the left half plane. In particular, any A-stable process with a Padé approximation $E(z)$ to $\exp(z)$, is well defined, and thus all processes considered in this paper are well defined.

2. Classes of A-Stable Processes.

Butcher's [2] v-stage processes of order $2v$, based on Gaussian quadrature formulae, have been shown to be A-stable by several authors (see [6] and [7]). Although Butcher's methods based on Radau and Lobatto quadrature formulae [3] are not A-stable, Ehle [6] has constructed similar methods which he conjectured to be A-stable. The A-stability of these processes is established in [4].

Table 2.1 summarizes some known classes of A-stable processes. The w_i and c_i are weights and abscissae of the appropriate quadrature formula, and $B = (b_{ij})$, $V = (c_i^{j-1})$, $C = (c_i^j/j)$, $N = (1/i)$ and $D = \mathrm{diag}(w_i)$ are all square matrices of dimension v.

Table 2.1. *A-Stable Runge–Kutta Processes.*

Class	Quadrature formula	B defined by	Order
G (Butcher)	Gaussian	$B = CV^{-1}$	$2v$
I_A (Ehle)	Radau, $c_1 = 0$	$B = D^{-1}(V^T)^{-1}(N - C)^T D$	$2v - 1$
II_A (Ehle)	Radau, $c_v = 1$	$B = CV^{-1}$	$2v - 1$
III_A (Ehle)	Lobatto	$B = CV^{-1}$	$2v - 2$
III_B (Ehle)	Lobatto	$B = D^{-1}(V^T)^{-1}(N - C)^T D$	$2v - 2$
III_C (Chipman)	Lobatto	Equation (2.1)	$2v - 2$

The collocation methods of Wright [7] and Axelsson [1] based on Gaussian, Radau and Lobatto quadrature formulae are equivalent to class G, II_A and III_B processes respectively.

In addition to Ehle's class III_A and III_B processes, a third class can be defined, based on Lobatto quadrature formulae. This class, referred to as class III_C, is defined in the following way:

1. Choose w_i and c_i as the weights and abscissae of a v-point Lobatto quadrature formula.
2. Choose $b_{i1}=w_1$, $i=1,2,\dots v$.
3. Choose the remaining b_{ij} by

$$\begin{pmatrix} b_{12} & \cdots & b_{1v} \\ \vdots & & \vdots \\ b_{v2} & \cdots & b_{vv} \end{pmatrix} = \begin{pmatrix} c_1-w_1 & \frac{c_1^2}{2} & \cdots & \frac{c_1^{v-1}}{v-1} \\ \vdots & \vdots & & \vdots \\ c_v-w_1 & \frac{c_v^2}{2} & \cdots & \frac{c_v^{v-1}}{v-1} \end{pmatrix} \begin{pmatrix} 1 & c_2 & \cdots & c_2^{v-2} \\ \vdots & \vdots & & \vdots \\ 1 & c_v & \cdots & c_v^{v-2} \end{pmatrix}^{-1}. \tag{2.1}$$

In [4] it is established that these processes are of order $2v-2$, and are strongly A-stable since $E(qh)$ in equation (1.2) becomes a second sub-diagonal Padé approximation to $\exp(qh)$. Similarly it is noted that processes in classes I_A and II_A are also strongly A-stable. Coefficients for several processes in class III_C are given in the appendix.

It is the author's experience that these methods can be used to solve stiff systems of ordinary differential equations at least as efficiently as methods now in use. A future paper will deal with the implementation of A-stable Runge-Kutta processes.

Appendix.

Coefficients for class III_C processes, $v \leqq 5$.

$v = 2$

$$\begin{array}{cc|c} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \hline \frac{1}{2} & \frac{1}{2} & \end{array}$$

$v = 3$

$$\begin{array}{ccc|c} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} & \frac{1}{2} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 1 \\ \hline \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \end{array}$$

$$v = 4 \quad \begin{array}{cccc|c}
\frac{1}{12} & -\frac{\sqrt{5}}{12} & \frac{\sqrt{5}}{12} & -\frac{1}{12} & 0 \\[2ex]
\frac{1}{12} & \frac{1}{4} & \frac{10-7\sqrt{5}}{60} & \frac{\sqrt{5}}{60} & \frac{5-\sqrt{5}}{10} \\[2ex]
\frac{1}{12} & \frac{10+7\sqrt{5}}{60} & \frac{1}{4} & -\frac{\sqrt{5}}{60} & \frac{5+\sqrt{5}}{10} \\[2ex]
\frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} & 1 \\[2ex]
\hline
\frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} &
\end{array}$$

$$\begin{array}{l} v = 5 \\[2ex] r = \sqrt{\frac{3}{7}} \end{array} \quad \begin{array}{ccccc|c}
\frac{1}{20} & -\frac{7}{60} & \frac{2}{15} & -\frac{7}{60} & \frac{1}{20} & 0 \\[2ex]
\frac{1}{20} & \frac{29}{180} & \frac{47-105r}{315} & \frac{29-30r}{180} & -\frac{3}{140} & \frac{1-r}{2} \\[2ex]
\frac{1}{20} & \frac{7}{180}\left(\frac{47+105r}{16}\right) & \frac{73}{360} & \frac{7}{180}\left(\frac{47-105r}{16}\right) & \frac{3}{160} & \frac{1}{2} \\[2ex]
\frac{1}{20} & \frac{29+30r}{180} & \frac{47+105r}{315} & \frac{29}{180} & -\frac{3}{140} & \frac{1+r}{2} \\[2ex]
\frac{1}{20} & \frac{49}{180} & \frac{16}{45} & \frac{49}{180} & \frac{1}{20} & 1 \\[2ex]
\hline
\frac{1}{20} & \frac{49}{180} & \frac{16}{45} & \frac{49}{180} & \frac{1}{20} &
\end{array}$$

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FACULTY OF MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO
CANADA

DEPT. OF MATHEMATICS
ACADIA UNIVERSITY
WOLFVILLE, NOVA SCOTIA
CANADA